

Longest-path attacks on complex networks

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We investigate the longest-path attacks on complex networks. Specifically, we remove approximately the longest simple path from a network iteratively until there are no paths left in the network. We propose two algorithms, the random augmenting approach (RPA) and the Hamilton-path based approach (HPA), for finding the approximately longest simple path in a network. Results demonstrate that steps of longest-path attacks increase with network density linearly for random networks, while exponentially increasing for scale-free networks. The more homogeneous the degree distribution is, the more fragile the network, which is totally different from the previous results of node or edge attacks. HPA is generally more efficient than RPA in the longest-path attacks of complex networks. These findings further help us understand the vulnerability of complex systems, better protect complex systems, and design more tolerant complex systems.

I. INTRODUCTION

Vulnerability is one of the fundamental properties in many nature and man-made complex systems[1–4]. For instance, genetic defects induce cell lesions[5], router failures interrupt the Internet[6], incidents or terrorist attacks cause collapse of public transport systems[7], and malfunction of a power station results in large-scale blackouts[8]. The vulnerability of complex systems is determined by the underlying networks in which nodes represent individuals in complex systems and edges represent the interactions of individuals. In past decades, many breakthroughs have been made on understanding individual components' disturbance's effects on the overall function of complex networks. Albert *et al*[9] found that the Internet and WWW (World Wide Web) are very robust against random failures, but quite fragile in intended attacks. This robust yet fragile property was confirmed in much larger maps of the WWW as well as many other scale-free networks[1], and percolation processes on model networks were introduced to explain this property[10, 11]. Since then many researchers studied the robustness of model networks and real-world networks subject to node attacks and edge attacks[1, 2], in which nodes or edges are removed in order of degree[12, 13] or other centrality measures such as betweenness[14, 15], eigenvector[16], PageRank[17], *etc.* A few localized failures or attacks may cause cascading failures and lead to breakdown of the whole system which has been observed in Internet[18, 19], power grids[20, 21], financial systems[22], *etc.* Recently, researchers studied robustness of temporal or time-varying networks by extending the measures of centrality and robustness with temporal properties[23, 24]. Furthermore, current attention focuses on the percolation processes on multiplex or interdependent networks which better model catastrophic events in power grids, transport systems and many other interdependent systems[8, 25, 26]. However, attacks are

not merely limited to node or edge attacks that have been widely studied in the literature. For instance, hurricanes always have a large-scale effect on the public transport networks, terrorists usually prefer much larger scale attacks, and drugs always affect many targets. Therefore, we need to understand the effect of attacks on larger parts of complex networks other than nodes and edges, like path attacks. A simple path composed of nodes and edges is a common subpart of a network. In this context, “path” always means “simple path” which indicates that a node appears at most once in a path. This restriction is consistent with the attacks problem. In a network, a straightforward measure of paths is path length. Therefore, a natural question is how removals of the longest paths affect the function of a network. Finding the longest paths in general graphs is a well-known NP-hard problem in the literature[27]. Many approximation algorithms are proposed to approximate the longest paths including color-coding method[28], divide and conquer approach[29], algebraic approach[30], *etc.* In this letter, we propose two algorithms, RPA and HPA, to find the approximately longest paths during the attack processes. The efficiency of these two algorithms in the attacks is determined by the decay rates of the largest components in networks. The robustness of model networks and real-world networks to longest-path attacks are reflected by the steps of the iterative attacks.

II. MODEL OF LONGEST-PATH ATTACKS

In a large network, there is usually a huge number of paths which is much larger than the number of nodes and edges. In some sense, paths can be thought of as the combinations of nodes and edges. When attacking a network, we prefer to remove the critical paths so that the removal of these paths significantly degenerate the function of the network. There are plenty of centrality measures in literature[1], but most of them are for nodes and edges, few are connected with paths. Path length, a natural measure for a path, is defined as the number of edges in a path. In our path-attack process, each time

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we remove the longest path from the network based on specified algorithms. Note that when attacking a path, we just remove all its edges from the network, but keep all its nodes in the network. The removal of longest path continues until there are no paths in the networks. In fact there are no edges in the final network, since edges are paths of length 1, and they would be removed from the network if there were any left.

III. MEASURES OF LONGEST-PATH ATTACKS

The size of the largest component in a network reflects the communication capability of a network. Therefore we compute the size of largest component S during each step of the longest-path attacks, which indicates the efficiency of an approximating algorithm. The larger the decay rate of the largest component, the more efficient the approximating algorithm. Also, we count the total step of the longest-path attacks T , which reflects the tolerance of a network subject to longest-path attacks, and more steps means a more robust network against longest-path attacks.

IV. ALGORITHM FOR APPROXIMATING THE LONGEST PATH

In each iteration of the attacks process, we need to find the longest path in the corresponding network, which is a famous NP-hard problem. Here we present two algorithms, random path augmenting approach (RPA) and Hamilton-path based augmenting approach (HPA), to find the approximate longest path in a network.

A. Random path augmenting approach

RPA is a very simple algorithm for constructing a path in a network. First, RPA randomly selects a node from the network as the root node. Second, RPA chooses a neighboring node of the root node as the second node of the path, and then chooses a neighboring node of the second node as the third node of the path, and so on. The path augmenting process continues until it cannot find a new appropriate node. The main procedure of RPA is as follows:

- (1) Maintain a path P which is a node sequence (initially null), and node set V_0 (initially null).
- (2) Randomly select a node r as the root node.
- (3) Append r to the end of P , and add r to V_0 .
- (4) Find a neighboring node of r named q , such that $q \notin V_0$.
- (5) IF q exists, take q as r , return to (3). Otherwise, the algorithm stops.

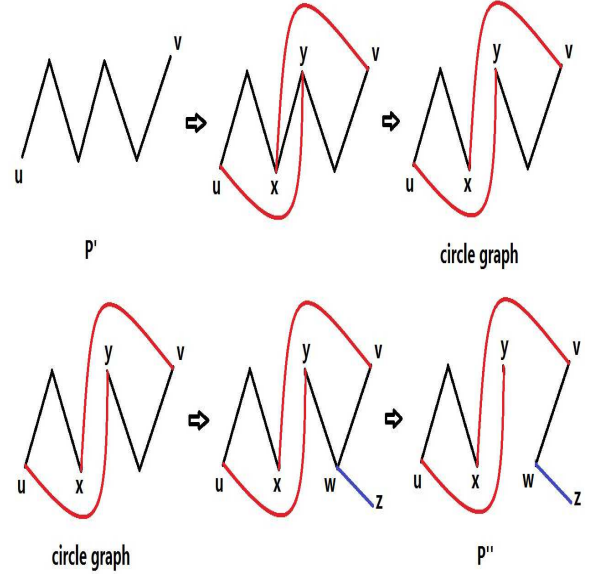


FIG. 1. An example for illustrating the idea of HPA.

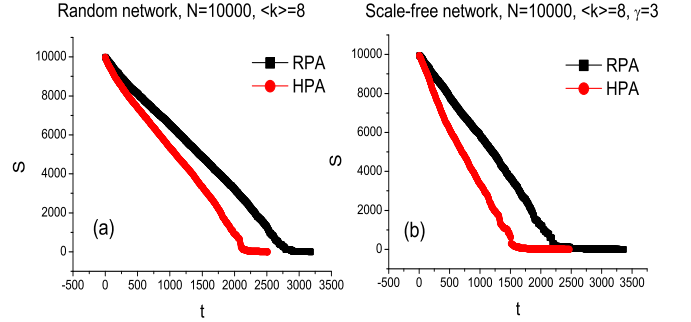


FIG. 2. Size of the largest component S vs. time step of the longest-path attack t for (a) the random network and (b) the scale-free network.

B. Hamilton-path based augmenting approach

HPA employs the idea of approximating the Hamilton path in a network[31]. Initially, HPA randomly selects a node as the root node, and augments a path as long as possible in both directions of the root node. In each iteration, HPA first generates a circle graph based on the current augmenting path. Then HPA makes the circle graph a new augmenting path by introducing an arbitrary node connected to the circle graph, and starts a new augmenting process. The procedure of HPA is as follows:

- (1) Maintain a path P that is a node sequence (initially null).
- (2) Randomly select a node r as the root node, and let $P = \langle r \rangle$.
- (3) Augment a path in both directions from the current path P as far as possible, and obtain a new

path P' . Assuming $P = \langle u0, \dots, v0 \rangle$, then $P' = \langle u, \dots, u0, \dots, v0, \dots, v \rangle$. Note that a node is allowed to appear once in a path.

- (4) Find an edge $\langle x, y \rangle$ in P' such that edge $\langle u, y \rangle$ and edge $\langle x, v \rangle$ exist. If edge $\langle x, y \rangle$ is found, then delete it, and add edge $\langle u, y \rangle$ and edge $\langle x, v \rangle$ to P' , which makes P' a circle graph, shown in the fig. 1. If edge $\langle x, y \rangle$ is not found, the algorithm ends, and P' is the resulting path. Note that if the network satisfies Dirac constraints and P' is not a Hamilton path, then $\langle x, y \rangle$ exists (see theorem 1 in Appendix A).
- (5) Find a node z , such that z is not included in the circle graph, but z connects to an arbitrary node w in the circle. If z is found, then delete an arbitrary edge incident to w from the circle graph, and add edge $\langle z, w \rangle$ to the circle graph. As a result, the circle graph becomes a longer path P'' , as shown in fig. 1. If z is not found, the algorithm ends, and P' is the resulting path. Note that if the network satisfies the Dirac constraints and P' is not a Hamilton path, then z exists (see theorem 2 in Appendix B).
- (6) Take P'' as P , and return to (3).

The circle graph and the augment path, generated in the intermediate stage of HPA, become larger with the increasing iteration times. We infer that if the network is dense enough (satisfying the Dirac constraints), then the resulting path explored by HPA is a Hamilton path, which is exactly the longest path in the network.

C. Space and time complexity of the two algorithms

Assume there are n nodes and m edges in a network. For one iteration, RPA runs in $O(n)$ space constantly and runs in $O(n)$ time in the worst case, while HPA runs in $O(n)$ space and $O(n^2)$ time in the worst case. For many iterations, RPA only runs in $O(m)$ time, while HPA only runs in $O(\lfloor m/n \rfloor * n^2)$ time, which are irrelevant with the specific iteration times. More detailed analysis of computing complexity for RPA and HPA is presented in Appendix C.

V. SIMULATION RESULTS

In this part, we show the results of longest-path attacks for model networks and real-world networks. The Erdős-Rényi (ER) model[32] and the static model[33] are employed to generate the random networks and scale-free networks respectively. The statistics of the real-world networks are shown in table. 1.

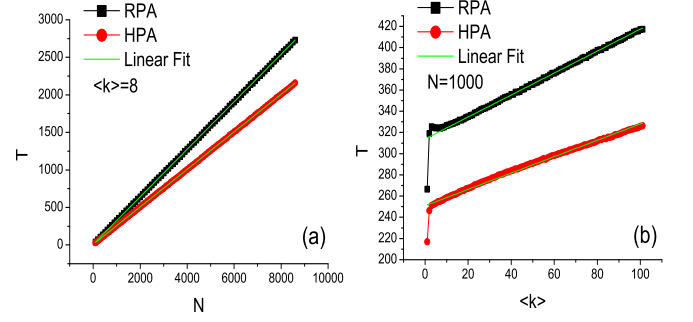


FIG. 3. (a) Total attack steps T vs. network size N , and (b) total attack steps T vs. average degree $\langle k \rangle$ for random networks. The results are the average of 10^4 independent runs.

A. Model networks

We first investigate decrease of size of the largest component S during the iterative longest-path attacks. In fig. 2, we clearly see that, S decreases with time step t almost linearly, and fast in the first stage, and slowly in the final stage. The behavior of S is similar in both the random network and the scale-free network. The efficiency of RPA and HPA is different which is inferred from the corresponding decay rates of S . HPA is more efficient than RPA since HPA has a much smaller S for a specific t , which is found in both the random network and the scale-free network.

Then we investigate the total attack steps T of model networks for the two algorithms, shown in fig. 3 and fig. 4. T increases linearly with network's size N for the two algorithms on both random networks (fig. 3(a)) and scale-free networks (fig. 4(a) and fig. 4(b)). For a given N , T of HPA is always smaller than that of RPA. T increases with average degree $\langle k \rangle$ linearly for random networks (fig. 3(b)), while exponentially for scale-free networks (fig. 4(c) and fig. 4(d)). However, when the power-law parameter $\gamma = 7$, T increases linearly as shown in fig. 4(c) and fig. 4(d). This is because the heterogeneity of the degree distribution of scale-free networks decrease with the power-law parameter γ [33]. Therefore, when γ is large enough, the scale-free networks are actually random networks. The increase of T with $\langle k \rangle$ indicates that the denser the network, the more robust the network is to longest path attacks. Also for specific $\langle k \rangle$, T of HPA is always smaller than that of RPA. From fig. 4(e) and fig. 4(f), we see that T decreases with the power-law parameter γ of scale-free networks. This means that the more homogeneous the degree distribution, the more fragile the network is to longest-path attacks, which is completely opposite to the previous results of node or edge attacks[4, 9]. Furthermore, we obtain from fig. 3 and fig. 4 that the networks are generally more robust to RPA than to HPA, which is inferred from the behavior of T , and holds for both random networks and scale-free networks.

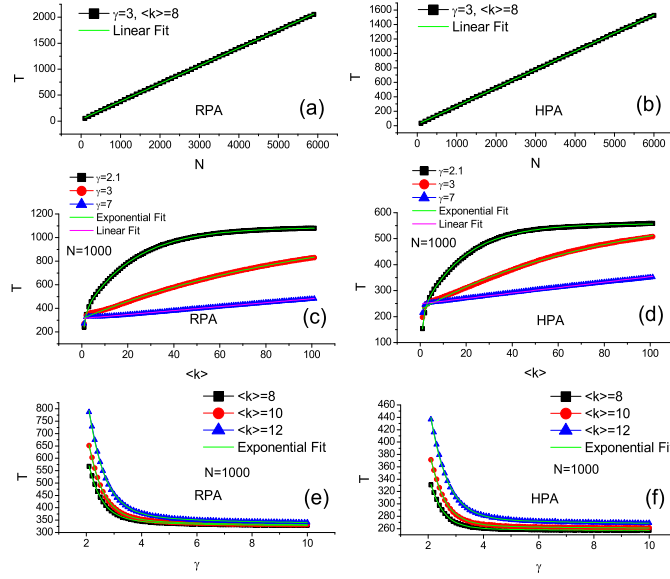


FIG. 4. Total attack steps T vs. network size N ((a) (b)), total attack steps T vs. average degree $\langle k \rangle$ ((c) (d)), total attack steps T vs. power-law parameter γ ((e) (f)) for scale-free networks. The results are the average of 10^4 independent runs.

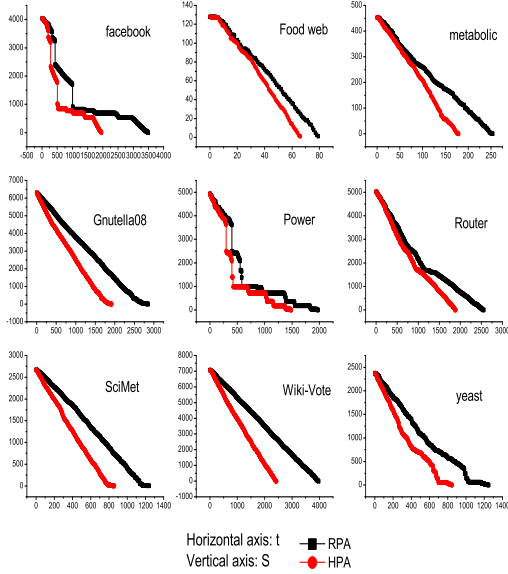


FIG. 5. Size of the largest component S vs. time step of the longest-path attack t for real-world networks.

B. Real-world networks

We also demonstrate the behavior of the largest component S in the longest-path attacks and the total steps of attacks T for real-world networks (table 1). In fig. 5, generally the size of the largest component S decreases with time step t linearly, which is similar in model networks. However, the fluctuations in the curves of some real-world networks are clear. HPA is more efficient than RPA for real-world networks, since HPA always has a much smaller S for a specified t shown in fig. 5. The ro-

TABLE I. Attack steps T for real-world networks.

NAME	NODES	EDGES	RPA	HPA
facebook	4039	88234	3486	1953
Food web	128	2106	79	66
Kohonen	4470	12731	1874	1214
metabolic	453	4596	254	179
Gnutella08	6301	20777	2853	1920
Power	4941	6594	1976	1487
Router	5022	6258	2555	1890
Wiki-Vote	7115	103689	3976	2435
Yeast	2375	11693	1250	848
SciMet	3084	10416	1240	858
USAir	1532	2126	233	147
polbooks	105	441	63	43
karate	34	78	22	14
Jazz	198	5484	112	82
C. elegans	297	2359	27	22
adjnoun	112	425	58	40

bustness of real-world networks to longest-paths attacks is reflected by the corresponding total steps of attacks T , shown in table 1. We obtain that real-world networks are more fragile to HPA than to RPA, since HPA has a much smaller T than RPA.

VI. CONCLUSIONS AND DISCUSSIONS

In summary, we investigate the robustness of complex networks, including model networks and real-world networks, which are subject to iterative attacks on the longest paths. Two algorithms are proposed to approxi-

mate the longest paths during the attacks. We compare the efficiency of the two algorithms by the decay rate of the largest components, and determine the robustness of the networks by the total attack steps in the longest-path attacks process. Specifically, we obtain that, total attack steps increase exponentially with network density for scale-free networks, while remaining linear for random networks. Total attack steps increase linearly with network size for both random networks and scale-free networks. For scale-free networks, the more heterogeneous the degree distribution, the more robust the networks are against longest-path attacks, which are totally different from the previous results of node or edge attacks. Note that finding the exact longest simple path in a general network is NP-hard. We only approximate the longest paths using two approximating algorithms. There should be more efficient approximating algorithms emerging in the future. We only investigate effects of the longest paths on the robustness of complex networks. However, our work may shed some light on the influence of longest path on the spread[34, 35], synchronization[36], control[37, 38], and other network dynamics.

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Appendix A: Theorem 1

Theorem 1: For any node s in the network, if $d(s) \geq \lceil |V|/2 \rceil$ (Dirac constraints), $d(s)$ is the degree of s , V is the node set of the network, then the edge $\langle x, y \rangle$ in step (4) of HPA exists.

Proof: According to the constraints, we have $d(u) \geq \lceil |V|/2 \rceil$, and $d(v) \geq \lceil |V|/2 \rceil$. Since u and v are the two end nodes of P' , which is the intermediate maximum augmenting path, all the neighbors of u and v should be on P' . Based on the inclusion-exclusion principle, we can easily obtain that edge $\langle x, y \rangle$ exists. The explanation is as follows:

- (1) If we take the node positions in P' as vacancies, then there are at most $|V|$ vacancies denoted by $a_1, a_2, \dots, a_{|V|}$. Once an arbitrary neighbor node of u takes one of the vacancies a_i , then a_{i-1} can not be occupied by any neighbor node of v . (Otherwise, nodes in a_{i-1} and a_i form the node pair $\langle x, y \rangle$ we need.)
- (2) Since $d(u) \geq \lceil |V|/2 \rceil$, the neighbor nodes of u take at least $\lceil |V|/2 \rceil$ vacancies. According to (1), there

are at least $\lceil |V|/2 \rceil$ vacancies which cannot be occupied by v 's neighbor nodes. On the other hand, the maximal number of vacancies for v 's neighbor nodes is $|V| - 1$ in theory. Therefore, the number of remaining vacancies for the neighbor nodes of v are smaller than $\lceil |V|/2 \rceil$. However, v 's neighbor nodes need no less than $\lceil |V|/2 \rceil$ vacancies, since $d(v) \geq \lceil |V|/2 \rceil$. This means that v 's neighbor nodes have to occupy at least one of the vacancies such as a_j , which is forbidden for v 's neighbor nodes. Then nodes in a_j and a_{j+1} form the node pair $\langle x, y \rangle$ we need.

Appendix B: Theorem 2

Theorem 2: For any node s in the network, if $d(s) \geq \lceil |V|/2 \rceil$ (Dirac constraints), $d(s)$ is the degree of s , V is the node set of the network, then the node z in step (5) of HPA exists.

Proof: Assume there is no such node z that connects the circle, then the circle is an independent subgraph of G (the whole network), which means the circle is separated from G -circle. Suppose there are n_0 nodes in the circle. For any node i in the circle, $d(i) \leq n_0 - 1$. For any node j in G -circle, $d(j) \leq (|V| - n_0) - 1$. Then $d(i) + d(j) \leq |V| - 2$, which means $d(i)$ or $d(j)$ is less than $\lceil |V|/2 \rceil$. This contradicts the constraints.

Appendix C: Analysis of space and time complexity

We assume that an undirected and unweighted graph is denoted by $G(V, E)$, V is the node set, and E is the edge set of the graph. Suppose $n = |V|$, $m = |E|$, and $p = \lceil m/n \rceil$ indicates the density of the graph. Obviously $0 \leq p \leq n$.

1) Random path augmenting approach

RPA is a simple algorithm for augmenting a path. Obviously it has $O(n)$ space complexity. For one iteration, RPA runs in $O(n)$ time, in the worst case. Also, the number of iterations of the longest-path attacks is at most $O(n)$. Let us assume the number of iterations is T , the length of the resulting path in the i -th iteration is A_i (Time complexity is $O(A_i)$). Since, $m = \sum_{i=1}^T A_i$, we obtain the total time complexity for RPA is $O(A_1) + O(A_2) + \dots + O(A_T) = O(m)$.

2) Hamilton-path based augmenting approach

HPA runs in $O(n)$ space in the worst case. Time complexity for one iteration is $O(n^2)$. The number of iterations is at most $O(m)$. Assume the number of iterations is T , the length of the explored path in the i -th iteration is A_i . Then the time complexity for the i -th iteration varies from $O(A_i)$ to $O(A_i^2)$. Since $m = \sum_{i=1}^T A_i$, we obtain that the overall time complexity TC that satisfies the following constraints of quadratic programming:

- (1) $0 < T \leq m$;

$$(2) \ 0 < A_i \leq n, i = 1, \dots, T;$$

$$(3) \ m = \sum_{i=1}^T A_i;$$

$$(4) \ TC = O(\sum_{i=1}^T A_i^2).$$

When the distribution of A_i is extremely uneven, HPA has the worst time complexity, which is calculated as follows:

$$\begin{aligned} TC &= O\left(\sum_{i=1}^T A_i^2\right) \\ &\leq O([m/n] * n^2 + (T - [m/n])) \\ &= O([m/n] * n^2) \\ &= O(p * n^2) \end{aligned} \quad (C1)$$

According to eq. C1, we obtain the following conclusions:

(1) The worst time complexity is irrelevant with the number of iterations of attacks, and it only depends on network scale and network density.

(2) The worst time complexity for one iteration is $O(n^2)$. For many iterations, the total time complexity grows at most by an order of $O(p)$ magnitude, obviously $O(p) \leq O(n) \leq O(m)$.

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